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Investigations on the applicability of fuzzy inference

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Abstract: This paper investigates the applicability of some fuzzy implication operators which were evaluated in the literature and are considered to be very good operators. We show that $(\wedge, \min(1, 1-\mu_A(a)+\mu_B(b)))$ has the best performance in the cases examined. Some new fuzzy implication operators are also investigated and found comparable in all cases to the operator given above.

Keywords: Fuzzy implication operator; fuzzy inference; fuzzy logic controller.

1. Introduction

Since the introduction of the basic methods of fuzzy reasoning by Zadeh [24] and the success of its first application, developed by Mamdani [13], fuzzy reasoning and its applications have been widely studied [1–27].

In analyzing the literature, it is clear that the selection of the best fuzzy implication operator is a fundamental problem in the design of fuzzy inferencing systems and fuzzy logic controllers. This subject has been discussed by many authors such as Bandler and Kohout [3, 4], Cao and Kandel [6], Fukami [8], Mizumoto [16–20], Kiszka [11, 12].

Fukami and Mizumoto [8, 18] investigated the inference results in the cases of generalized modus ponens and generalized modus tollens using several fuzzy implication operators. From their investigations, they concluded that R_5 and R_8 (the definition of all operators discussed in this paper can be found in Table 1.1) were not suitable for the fuzzy reasoning under certain criteria. Furthermore, they suggested several new fuzzy implication operators satisfying their intuitions such as R_2. Later, however, Mizumoto [19] showed that the inference results inferred by R_5 did coincide with previous intuition with respect to several criterial not using the max–min composition but using a new composition as the compositional rule of inference.

Kiszka [11, 12] investigated seventy-two fuzzy implication operators on the accuracy of a fuzzy model of a d.c. series motor. R_2 and R_{27} were considered the best fuzzy implication operators since these operators allowed the least root-mean-square error with the minimal number of mathematical operations necessary for computer implementation in their investigations. Their investigations were based on only one typical example (i.e., d.c. series motors) as well as one typical verbal description such as ZERO, SMALL, . . . , LARGE of the variable, I, and SMALL, MEDIUM, . . . , LARGE of the variable, N. In addition, the membership functions of each verbal description for I and N were fixed and rather simple.

Mizumoto [20] also compared control results for a plant model with first order delay using various fuzzy implication operators and showed that R_8 had the best result. Even though he investigated how control results were influenced when fuzzy sets of linguistic control rules were changed, he used only one typical example (i.e., plant model) with first order delay and one typical control rule and changed only the width of fuzzy sets of linguistic control rules.

On the other hand, Cao and Kandel [6] discussed the performance of seventy-two

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Table 1.1. A list of fuzzy implication operators, u is a particular value of the variable A and v is a particular value of the variable B and $\mu_A(u)$ and $\mu_B(v)$ are the grade of membership

$$R_2 = \int_{U\times V} [\mu_A(u) \rightarrow_2 \mu_B(v)]/(u, v)$$

$$\text{where } \mu_A(u) \rightarrow_2 \mu_B(v) = \begin{cases} 1 & \text{if } \mu_A(u) \le \mu_B(v) \\ 0 & \text{otherwise} \end{cases}$$

$$R_5 = \int_{U\times V} [1 \wedge (1 - \mu_A(u) + \mu_B(v))]/(u, v)$$

$$R_8 = \int_{U\times V} [\mu_A(u) \wedge \mu_B(v)]/(u, v)$$

$$R_{22} = \int_{U\times V} [1 \wedge (1 - \mu_A(u) + \mu_A(u) \times \mu_B(v))]/(u, v)$$

$$R_{25} = \int_{U\times V} [\mu_A(u) \times \mu_B(v)]/(u, v)$$

$$R_{27} = \int_{U\times V} [\mu_A(u) \rightarrow_{27} \mu_B(v)]/(u, v)$$

$$\text{where } \mu_A(u) \rightarrow_{27} \mu_B(v) = \begin{cases} 1 & \text{if } \mu_A(u) \le \mu_B(v) \\ \mu_B(v) & \text{otherwise} \end{cases}$$

$$R_{31} = \int_{U\times V} [0 \vee (\mu_A(u) + \mu_B(v) - 1)]/(u, v)$$

$$R_a = \int_{U\times V} [1 \wedge (1 - (\mu_A(u) - \mu_B(u))^2)]/(u, v)$$

$$R_b = \int_{U\times V} [1 \wedge (1 - |\mu_A(u) - \mu_B(v)|)]/(u, v)$$

implication operators in several different examples with different membership functions. In order to evaluate the applicability of fuzzy implication operators, they used the following three criteria:

(1) The maximal error caused by the implication operator must be within the range of the maximal error caused by the verbal descriptions.

(2) An implication operator with good applicability should be applied to the more practical cases.

(3) The results of fuzzy inference using the implication operator must be robust with regards to the choice of membership functions.

According to these criteria, they found that the operators $R_5, R_8, R_{22}, R_{25},$ and R_{31} were very applicable in all cases discussed in [6].

Based on the previous research [6], in this paper, we investigated the above implication operators as well as R_2 which Kiszka [11, 12] and other researchers preferred in fuzzy reasoning. In order to avoid losing the generality of a fuzzy model, we choose the simplest functional relation $Y = X$ as our model. In other words, we investigate fuzzy implication operators in $Y = X$. In addition to the above criteria, the result of fuzzy implication using the implication operator must be robust with regards to the choice of verbal statements and definition of membership functions. It should be suitable for the fuzzy implication operator to modify the shape of membership functions in order to get the result of fuzzy inference as accurate as possible. These are the criteria we will use in this paper.

In Section 2, we show that the applicability of R_2 is not robust in the choice of membership functions even in the simplest functional relation $Y = X$. In Section 3, we show that the generated result via the use of $R_8, R_{25},$ and R_{31} is unreasonable in $Y = X$ for each of the following cases:

Case 1. We define each membership function in more detail.

Case 2. We include more verbal descriptions.

Case 3. We add some verbal statement such as "IF X is not very small THEN Y is not very small" in the given verbal descriptions.

We show the fuzzy implication operator R_5 has very good performance in $Y = X$. For the purpose of seeing how well it applies to the practical cases, we investigated the applicability of the fuzzy implication operator R_5 on various examples such as $10^* \ln(1+x)/\ln(11)$, and a more complicated curve in Section 4. Furthermore, we introduce several new fuzzy implication operators which perform as well as R_5 in all cases discussed in this paper. Finally, Section 6 provides a conclusion of our investigation.

Throughout this paper, we adopt the following fuzzy conditional statements to describe a certain situation:

$$\begin{array}{l} \text{IF } X \text{ is } A(1) \text{ THEN } Y \text{ is } B(1) \\ \quad \text{ALSO} \\ \text{IF } X \text{ is } A(2) \text{ THEN } Y \text{ is } B(2) \\ \quad \vdots \\ \text{IF } X \text{ is } A(n) \text{ THEN } Y \text{ is } B(n) \end{array} \tag{1.1}$$

where X and Y are two variables, and $A(1), \ldots, A(n)$ and $B(1), \ldots, B(n)$ are verbal descriptions of the variable X and Y, respectively which can be quantified by fuzzy subsets A_i^* and B_i^* for each i in the ranges of variables X and Y. The above fuzzy conditional statements (1.1) can be formalized in the form of a fuzzy relation $R(x, y)$ [24]:

$$R(x, y) = R_1 \cdot R_2 \cdot \cdots \cdot R_n,$$

where operator $\cdot$ is the composition operator of two fuzzy relations and R_i denotes the fuzzy relation between X and Y determined by the i-th fuzzy conditional statement. That is

$$R_i = \int \mu_{Ai}(x) * \mu_{Bi}(y)/(x, y), \quad i = 1, 2, \ldots, n.$$

This pair of operators $\cdot$ and $*$ is a fuzzy implication operator. In order to calculate the deterministic value of a linguistic variable Y, which is defined as a variable whose value is a sentence in a natural or artificial language, the following defuzzification method is applied [11, 12]:

$$y = \sum_{k=1}^{n} y_k/n, \tag{1.2}$$

where y is a particular value of the variable Y and y_k are the support values in which the membership function $\mu_y(y)$ reaches its maximum grade of membership, and n is the number of support elements in which $\mu_y(y)$ reaches the maximum value, and k is an index from 1 to n.

We list several fuzzy implication operators including new operators R_a and R_b in Table 1.1 which will be used throughout this paper. For the fuzzy implication operators R_8, R_{25}, and R_{31} in Table 1.1, the sentence connective ALSO from the verbal description is interpreted as union ($\vee$ or maximum). On the other hand, for the fuzzy implication operators R_2, R_5, R_{22}, R_a, and R_b, the sentence connective ALSO from the verbal description is interpreted as intersection ($\wedge$ or minimum).

2. Influence of membership functions on the applicability of fuzzy implication operators

We investigate the influence of the membership functions on the applicability of the fuzzy implication operator R_2 in $Y = X$.

For simplicity, suppose we have three verbal descriptions for both variables X and Y, i.e., SMALL, MEDIUM, and LARGE. Thus, for the function $Y = X$, the following group of simple fuzzy conditional statements can be applied to describe this functional relation $Y = X$:

IF X is small THEN Y is small

ALSO

IF X is medium THEN Y is medium (2.1)

ALSO

IF X is large THEN Y is large

In this case, where the same verbal descriptions of variables X and Y have the same membership functions, the generated relation matrix is an identity matrix. In other words, we have exactly the same result curve as $Y = X$ by using the fuzzy implication operator R_2. However, the objects discussed in fuzzy conditional statements are distinct and the membership functions of their verbal descriptions are defined subjectively in most cases. That is, for each variable X and Y, the definition of membership functions is carried out independently. Therefore, it is reasonable that we discuss the case where the corresponding verbal descriptions of X and Y have different membership functions. Even for the function $Y = X$, we still consider that the corresponding membership functions of X and Y may be different. Thus the corresponding membership functions of X and Y may be subtly different in the case of $Y = X$. Of course, if we still describe the functional relation $Y = X$ by means of the fuzzy conditional statements (2.1) for different membership functions, then it is only an approximate description of $Y = X$. Therefore, for each value x and the fuzzy inference result y^*, it should be stated $y^* \approx x$.

To investigate the influence of choice of membership function for verbal descriptions, first we choose two groups of membership functions to quantify the verbal descriptions (see Figure 2.1 and Tables 2.1 and 2.2). In Figure 2.1, the straight line is for the verbal descriptions of X such as SMALL, MEDIUM, and LARGE and the - + - line is for the verbal descriptions of Y. With the different membership functions, the generated result is given in Figure 2.2. It should be noted that the difference of the corresponding membership degrees between verbal descriptions for X and Y is no more than 0.01. However, we have the result which is quite different from the $Y = X$ with slightly different membership func-

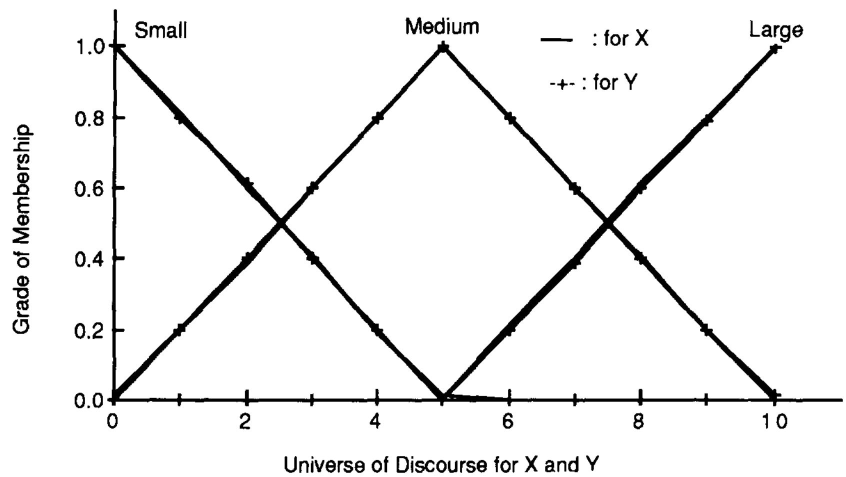

Fig. 2.1. The membership functions of fuzzy sets for X and Y with slightly difference.

Table 2.1. Fuzzy sets for X on $Y = X$

Universe	Small	Medium	Large
0	1.00	0.01	0.00
1	0.81	0.20	0.00
2	0.60	0.39	0.00
3	0.41	0.60	0.00
4	0.19	0.80	0.00
5	0.01	1.00	0.00
6	0.00	0.80	0.21
7	0.00	0.60	0.40
8	0.00	0.41	0.61
9	0.00	0.20	0.80
10	0.00	0.00	1.00

Table 2.2. Fuzzy sets for Y on $Y = X$

	Y		
Universe	Small	Medium	Large
0	1.00	0.00	0.00
1	0.80	0.20	0.00
2	0.61	0.40	0.00
3	0.40	0.60	0.00
4	0.20	0.80	0.00
5	0.00	1.00	0.00
6	0.00	0.80	0.20
7	0.00	0.60	0.39
8	0.00	0.40	0.60
9	0.00	0.20	0.79
10	0.00	0.01	1.00

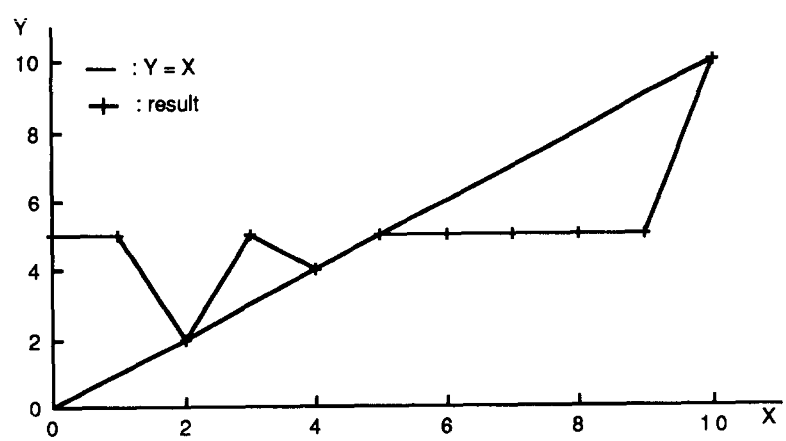

Fig. 2.2. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_2 with slightly different membership functions.

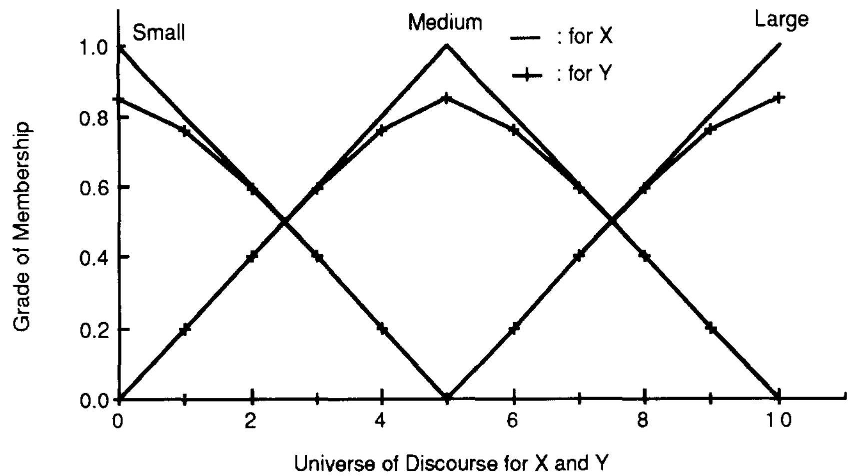

Fig. 2.3. The membership functions of fuzzy sets for X and Y when membership functions for X are higher than those for Y.

tions. The reason R_2 generates a considerably different result with slight variation in membership functions is that R_2 is too sensitive. For example, we have $0.51 \bullet 0.5 = 0$ and $0.5 \bullet 0.5 = 1$, where $\bullet$ is R_2, i.e., $x \bullet y = 1$ if $x \leq y$ and $x \bullet y = 0$ otherwise, even though the difference between 0.51 and 0.5 is 0.01.

Now, we choose the following two groups of membership functions of the verbal descriptions for X and Y, respectively (see Figure 2.3). In Figure 2.3 the grades of membership for the verbal descriptions of X is a little higher than those of Y. With the above ones, we have the following result which is far from $Y = X$ when you use R_2 (see Figure 2.4). The generated relation matrix is a zero matrix. That means there is no information at all. Since we take the average of the support values in which the membership function $\mu_Y(y)$ reaches its maximum grade of membership, we have the functional relation $Y = 5$ as a result instead of $Y = X$.

However, if we choose a membership function for X and Y (see Figure 2.5) in which the grades of membership of the verbal descriptions for Y is a little higher than those for X, we have exactly the same result $Y = X$. For the fuzzy implication operator R_2, $x \bullet y \neq y \bullet x$ where $x \bullet y = 1$ if $x \leq y$

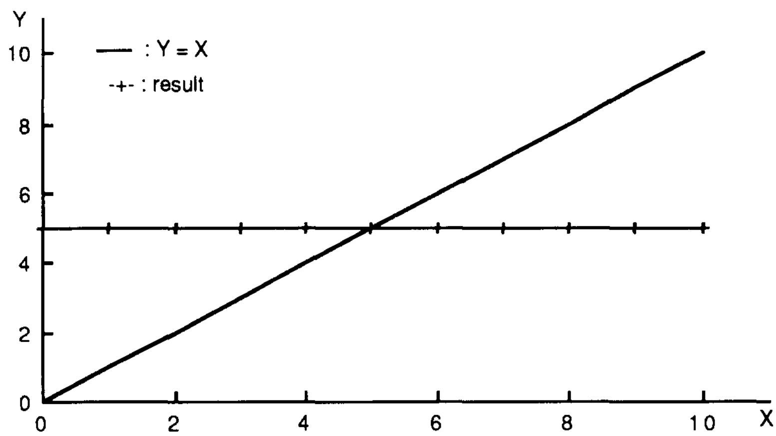

Fig. 2.4. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_2 when membership functions for X are higher than those for Y.

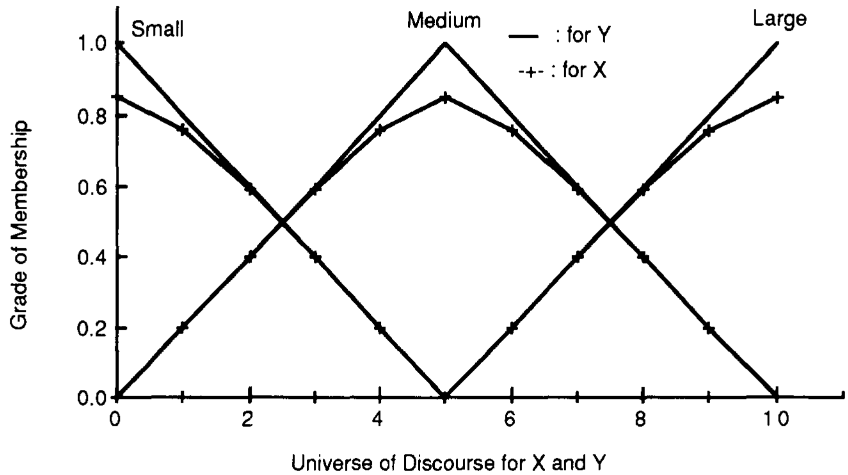

Fig. 2.5. The membership functions of fuzzy sets for X and Y when membership functions for Y are higher than those for X.

and $x \bullet y = 0$ otherwise and the sentence connective ALSO from verbal description is interpreted as intersection ($\wedge$ or minimum). For example $0.51 \bullet 0.5 = 0$ and $0.5 \bullet 0.51 = 1$; furthermore,

$$R' = (0.8/1 + 0.2/2 + 0.0/3) \bullet (0.76/1 + 0.2/2 + 0.0/3) = 0$$

and

$$R'' = (0.76/1 + 0.2/2 + 0.0/3) \bullet (0.8/1 + 0.2/2 + 0.0/3) = 1.$$

Therefore in the case of Figure 2.3, the generated relation matrix is a zero matrix and in the case of Figure 2.5, the generated relation matrix is an identity matrix. As a result, in order to get the correct result in $Y = X$, it is required for R_2 to have (1) exactly the same membership functions for X and Y or (2) the membership functions for Y higher than those for X. However, the choice of the membership functions for X and Y may be obtained independently. Furthermore, the process operator may choose different membership functions even for the same object [6]. Thus this requirement is not reasonable. On the other hand, changes of the membership functions do not affect the result inferred using the fuzzy implication operator R_5. We show that the result of fuzzy implication using the implication operator R_5 is robust with regards to the choice of membership functions.

3. Investigation of applicability of some fuzzy implication operators

In this section, we investigate the applicability of the fuzzy implication operators such as R_8, R_{25}, and R_{31} in $Y = X$. We show that the inference result via the use of these fuzzy implication operators is unreasonable in the following cases:

Case 1. When we define each membership function in more detail.

Case 2. When we include more verbal descriptions to describe the functional relation

$Y = X$.

Case 3. When we add some verbal statements such as "IF X is not very small THEN Y is not very small".

We discuss these three cases in Sections 3.1, 3.2 and 3.3, respectively.

We show that the fuzzy implication operators R_8, R_{25}, and R_{31} do not perform well in $Y = X$ despite their applicability in all cases discussed in [6]. However the fuzzy implication operator R_5 performs well in the above three cases.

3.1. More detailed definition of membership functions

We adopt the group of fuzzy conditional statements (2.1) to describe the functional relation $Y = X$ and investigate the generated result via the use of the fuzzy implication

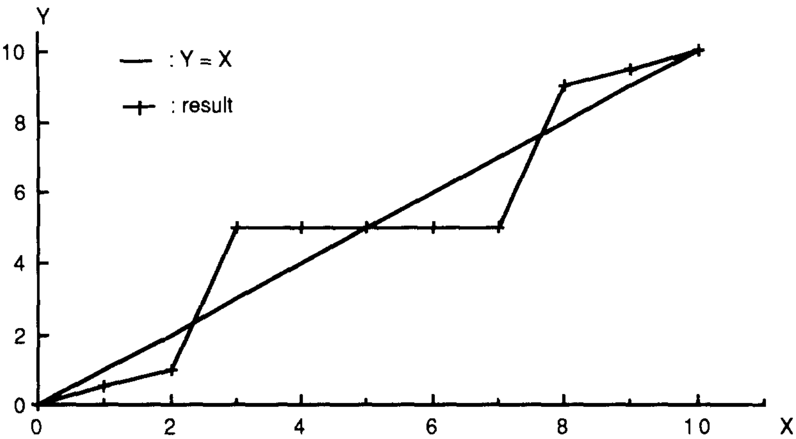

Fig. 3.1. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_8 with 11 points (average of error is 0.8182).

operators R_8, R_{25}, and R_{31} in the following cases: With fixed universe of discourse, i.e., $U = [0, 10]$, we define each of the fuzzy sets in (2.1) as follows:

(1) when $y_1 = 0, y_2 = 1, \ldots, y_{n-1} = 9, y_n = 10$, i.e., 11 points,

(2) when $y_1 = 0, y_2 = 0.5, \ldots, y_{n-1} = 9.5, y_n = 10$, i.e., 21 points

(3) when $y_1 = 0, y_2 = 0.25, \ldots, y_{n-1} = 9.75, y_n = 10$, i.e., 41 points.

These results are given in Fig. 3.1–Fig. 3.6. The reader can see that as the check points increase, the result is getting worse. In fact, as the check points increase, the average of error is higher. It is obviously unreasonable. However, the result of fuzzy inference using R_5 is accurate all the time in the above 3 cases. Thus we show that R_5 is robust with regards to the number of check points.

3.2. More verbal descriptions

Generally, the model result should be more accurate if we use more fuzzy subsets to describe a certain situation. The following describes the fuzzy subset used in three test cases for the relation $X = Y$:

(1) We use three fuzzy subsets – SMALL, MEDIUM, and LARGE. The result is given in Figure 3.1 for R_8 and Figure 3.4 for R_{25} and R_{31}.

(2) We use five fuzzy subsets – VERY SMALL, SMALL, MEDIUM, LARGE, VERY LARGE. The result is given in Figure 3.10 for R_8 and Figure 3.12 for R_{25} and R_{31}.

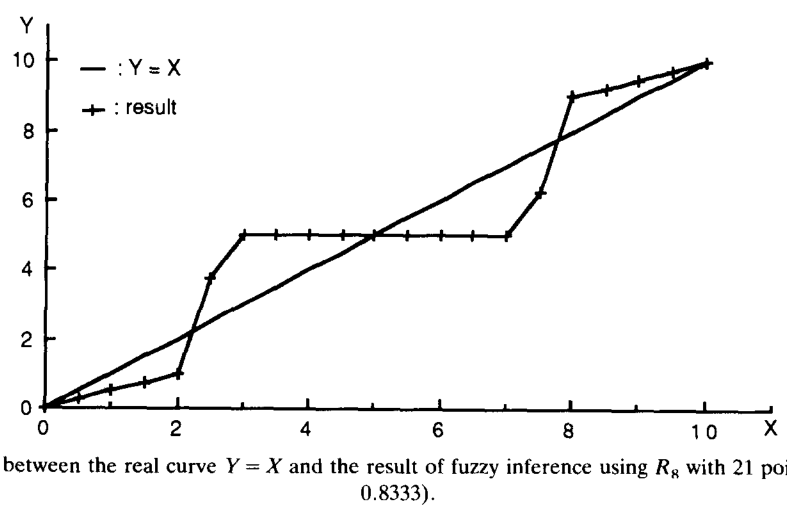

Fig. 3.2. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_8 with 21 points (average of error is 0.8333).

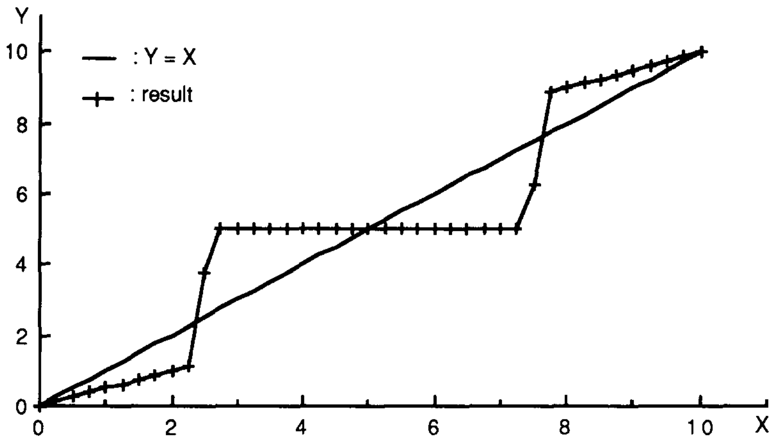

Fig. 3.3. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_8 with 41 points (average of error is 0.8841).

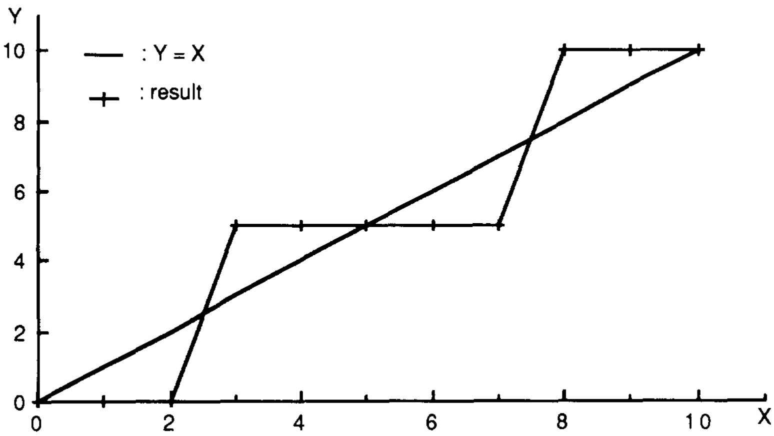

Fig. 3.4. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_{25} and R_{31} with 11 points.

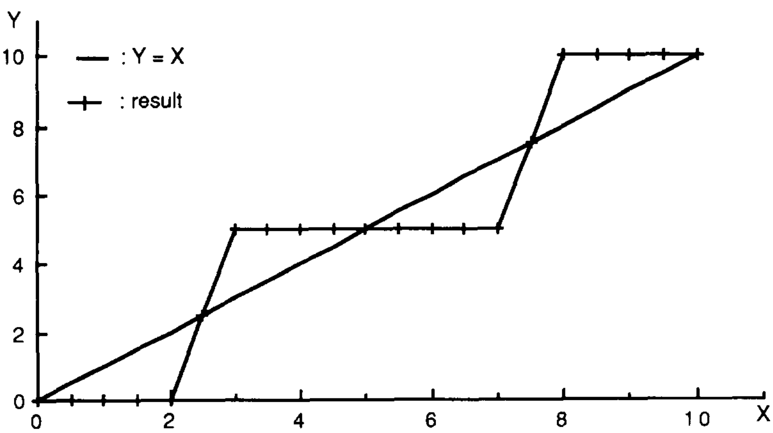

Fig. 3.5. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_{25} and R_{31} with 21 points.

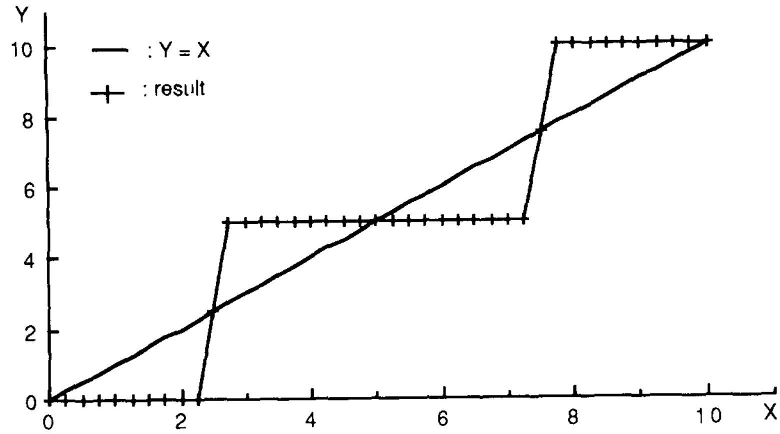

Fig. 3.6. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_{25} and R_{31} with 41 points.

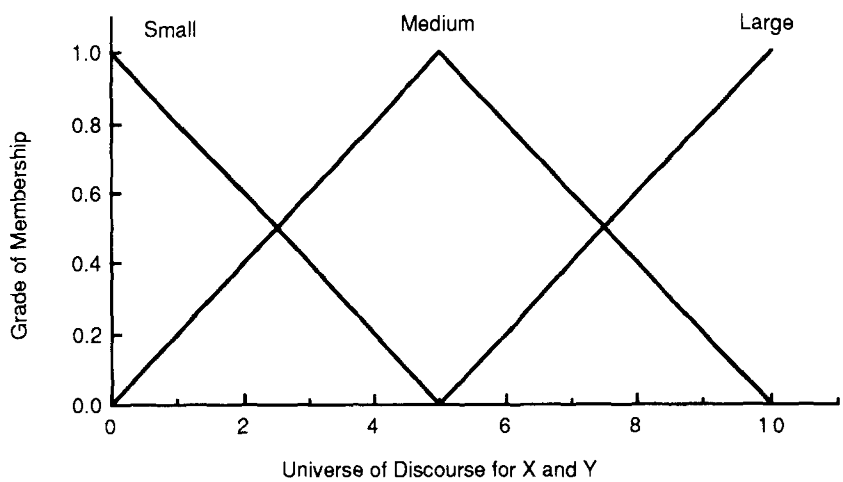

Fig. 3.7. The membership functions of fuzzy sets for X and Y (three verbal statements).

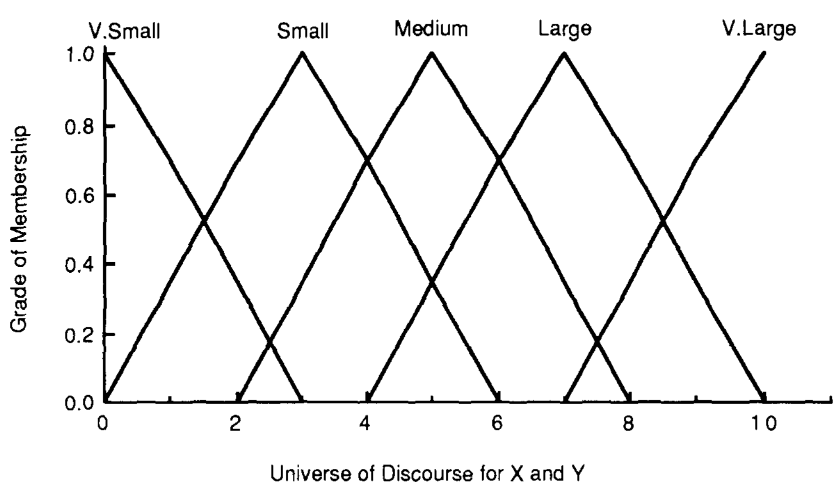

Fig. 3.8. The membership functions of fuzzy sets for X and Y (five verbal statements).

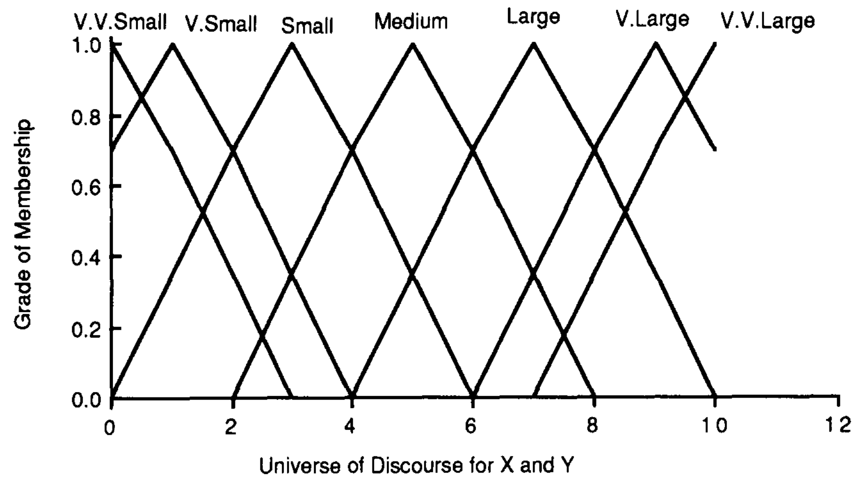

Fig. 3.9. The membership functions of fuzzy sets for X and Y (seven verbal statements).

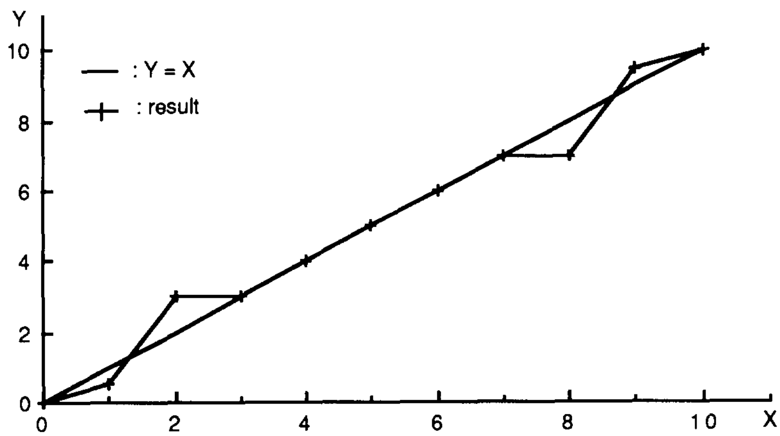

Fig. 3.10. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_8 with five verbal statements.

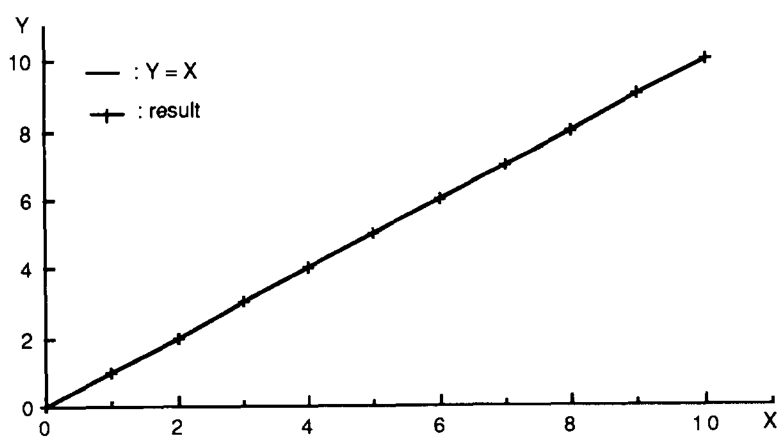

Fig. 3.11. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_8 with seven verbal statements.

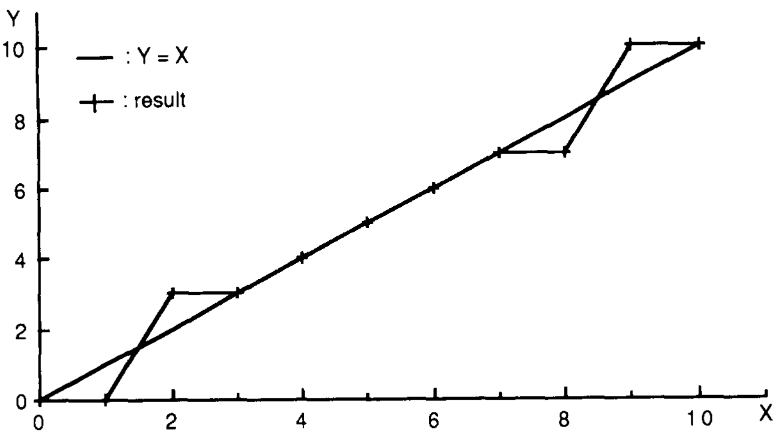

Fig. 3.12. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_{25} and R_{31} with five verbal statements.

(3) We use seven fuzzy subsets – VERY VERY SMALL, VERY SMALL, SMALL, MEDIUM, LARGE, VERY LARGE, VERY VERY LARGE. The results are given in Figure 3.11 for R_8 and Figure 3.13 for R_{25} and R_{31}. In addition, membership functions for each cases are given in Figures 3.7, 3.8 and 3.9 respectively.

Comparing the three experiments, the more fuzzy subsets used, the better the result obtained. The reason is largely due to the fact, by using more fuzzy subsets, we are able to provide more detailed information about the two relationships between X and Y. When we use seven fuzzy subsets to describe the functional relation $Y = X$, in case of operator R_8, R_{25}, and R_{31}, we got very accurate result (see Figures 3.11 and 3.13). But it is very interesting that the generated result is not perfect. In Figure 3.11, the universe of discourse is quantified from 0 to 10. If we define each fuzzy subset with 21 points then the obtained result is not the same as $Y = X$ (see Figure 3.14). it is similar but not identical. The reader can see that the result with 21 check points is not accurate via the use of R_8, R_{25}, and R_{31} even though it was perfect with 11 check points (see Figures 3.14 through 3.17). However, it should be noted that the result of fuzzy inference using R_5 is accurate in the 3 cases discussed in this section. Thus we show that R_5 is robust with regards to the number of verbal descriptions.

3.3. *Adding a verbal statement*

In order to describe $Y = X$, we now choose the following group of fuzzy conditional

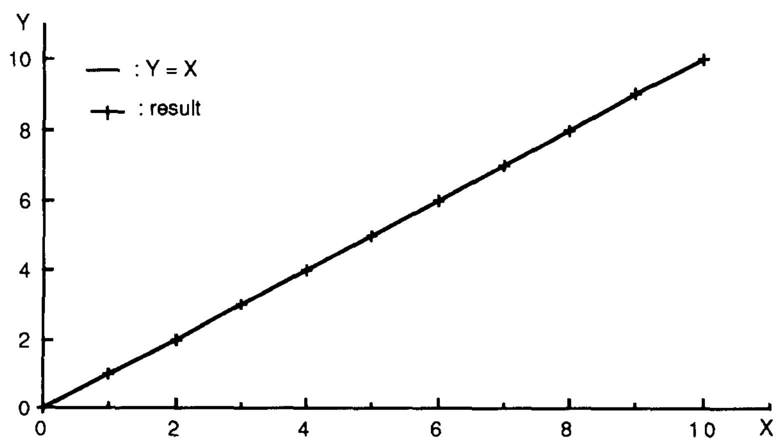

Fig. 3.13. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_{25} and R_{31} with seven verbal statements.

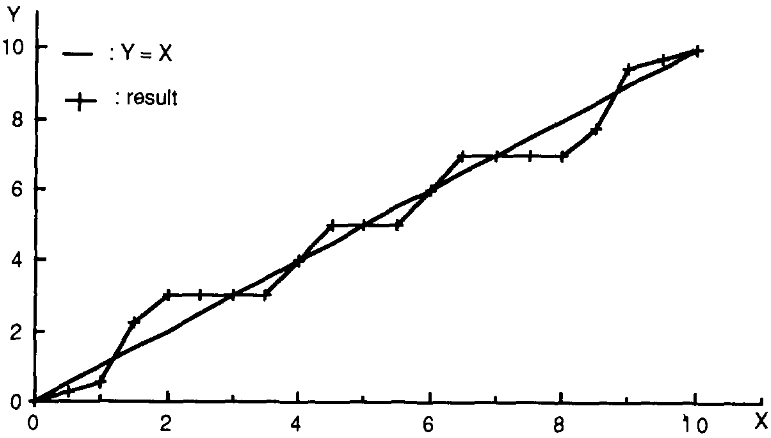

Fig. 3.14. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_8 with five verbal statements and 21 points.

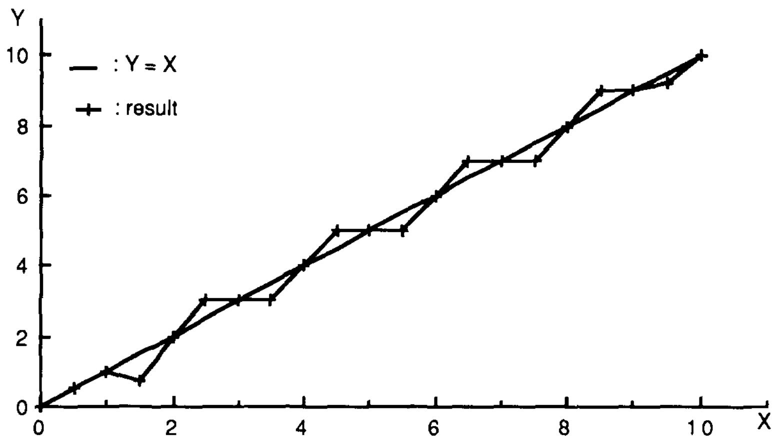

Fig. 3.15. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_8 with seven verbal statements and 21 points.

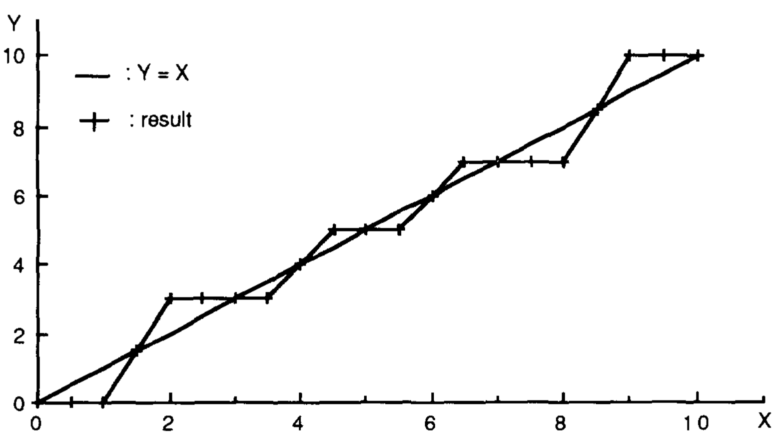

Fig. 3.16. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_{25} and R_{31} with five verbal statements and 21 points.

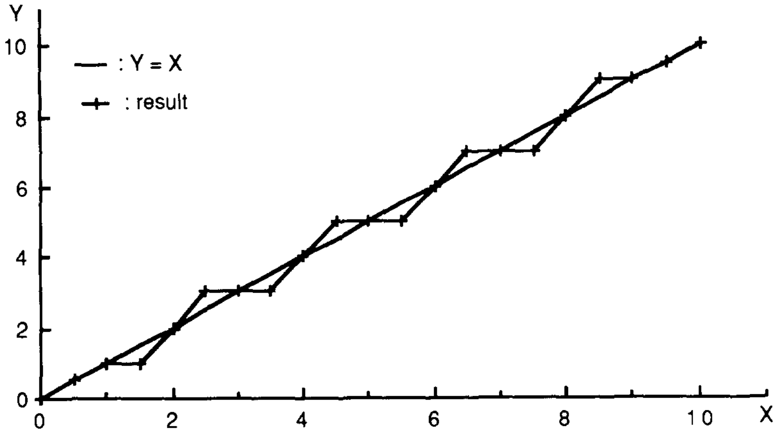

Fig. 3.17. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_{25} and R_{31} with seven verbal statements and 21 points.

statements:

IF X is very very small THEN Y is very very small

ALSO

IF X is very small THEN Y is very small

ALSO

IF X is small THEN Y is small

ALSO

IF X is medium THEN Y is medium (3.1)

ALSO

IF X is large THEN Y is large

ALSO

IF X is very large THEN Y is very large

ALSO

IF X is very very large THEN Y is very very large

Using seven fuzzy conditional statements, we have almost the same results as $Y = X$ which is given in Figure 3.11, and Figure 3.13 for the fuzzy implication operators R_8, and R_{25} respectively. One may expect that the more fuzzy conditional statements used, the better the result obtained. However, if we add the following fuzzy conditional statement in (3.1):

"IF X is not very small THEN Y is not very small",

which is an acceptable verbal description to describe $Y = X$, then the result is not reasonable (see Figures 3.18 and 3.19). However, it should be noted that the result inferred using R_5 is not affected when the above fuzzy conditional statement is added. Therefore, the result of fuzzy inference using R_5 is robust in the choice of verbal statements.

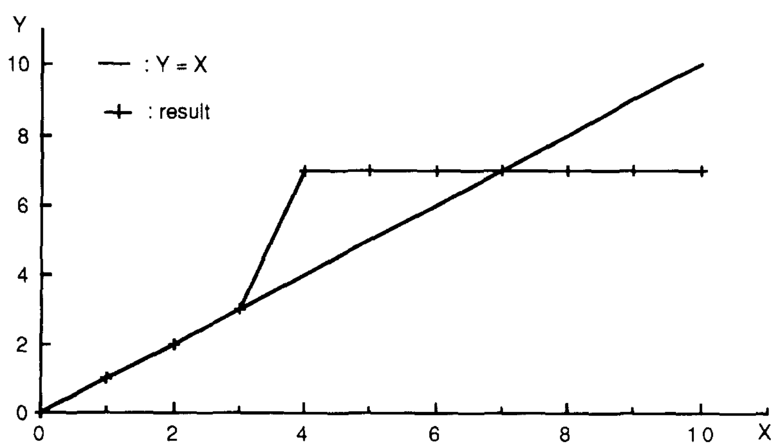

Fig. 3.18. Comparison between the real curve $Y = X$ and the result of fuzzy inference using R_8, R_{25}, and R_{31} when "If X is not very small then Y is not very small" is added.

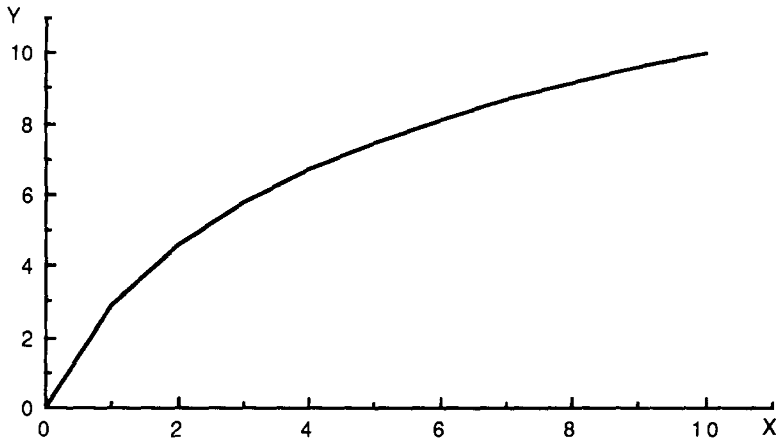

Fig. 4.1. The real curve of $10*\ln(1+x)/\ln(11)$.

4. Applicability of the fuzzy implication operator R_5

We have shown that the fuzzy implication operator R_5 performs well in all cases discussed in $Y = X$. Now we investigate the applicability of the fuzzy implication operator R_5 on the function $10*\ln(1+x)/\ln(11)$, and the more complicated curve in Sections 4.1 and 4.2, respectively. Moreover, we introduce new fuzzy implication operators R_a and R_b which are as good as R_5 in all cases discussed in this paper.

The fuzzy implication operator R_5 was proposed by Zadeh [24]:

$$R_5(A(x), B(y)) = (\neg A \times V) \bullet (U \times B)$$

$$= \int_{U \times V} 1 \wedge (1 - \mu_A(u) + \mu_B(v))/(u, v) \qquad (4.1)$$

where $\times$, $\bullet$, and $\neg$ denote Cartesian product, bounded-sum, and complement, respectively. For the fuzzy implication operator R_5, the sentence connective ALSO from the verbal description is interpreted as intersection ($\wedge$ or minimum).

4.1. Applicability in simulating a functional relation

We simulate the function $10*\ln(1+x)/\ln(11)$, where x is the real number which is from 0 to 10, using the fuzzy implication operator R_5. For the sake of convenience, we use only 11 numbers which are 0 through 10 for the variable x. The curve is given in Figure 4.1.

We use the following simple group of fuzzy conditional statements for the curve of $10*\ln(1+x)/\ln(11)$. One may use more fuzzy

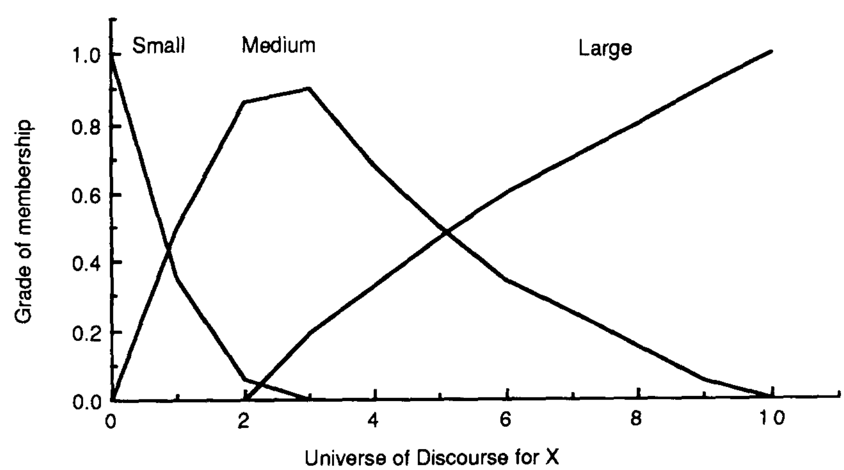

Fig. 4.2. The membership functions of fuzzy sets for X on $10*\ln(1+x)/\ln(11)$.

Table 4.1. Fuzzy sets for X on $10*(1+x)/\ln(11)$

Universe	Small	Medium	Large
0	1.00	0.00	0.00
1	0.35	0.50	0.00
2	0.06	0.86	0.00
3	0.00	0.90	0.19
4	0.00	0.68	0.33
5	0.00	0.50	0.47
6	0.00	0.34	0.60
7	0.00	0.25	0.70
8	0.00	0.15	0.80
9	0.00	0.05	0.90
10	0.00	0.00	1.00

conditional statements since the more fuzzy subsets used the better the result obtained.

IF X is small THEN Y is small

ALSO

IF X is medium THEN Y is medium (4.2)

ALSO

IF X is large THEN Y is large

As a matter of fact, the verbal description represented in (4.2) is exactly the same as the verbal description in (2.2) for $Y = X$. However, it should be noted that the meanings of the verbal description are different from those of the verbal description in (2.2) for $Y = X$. In other words, to make a fuzzy curve as accurate as possible, the membership function for X is manipulated. Table 4.1 is the meaning of membership functions for X (see Figure 4.2). The membership functions for Y are given in Figure 4.3. Using the above data, we find that the generated result curve is very close to the real curve of $10*\ln(x+1)/\ln(11)$ (see Figure 4.4).

4.2. Applicability in a more complicated curve

We simulate more complicated curves using the fuzzy implication operator R_5. Suppose we have the following curve shown in Figure 4.5 (see Table 4.2). As can be seen, the curve increases slowly between $x = 0$ and $x = 10$ and increases rapidly between $x = 10$ and $x = 15$ and its slope is close to 1 between $x = 15$ and $x = 20$.

We use the simple group of fuzzy conditional statements (4.2) for the complicated curve even if one may use more fuzzy conditional statements. However, the meanings of the verbal description are different (see Table 4.3 and Figures 4.6 and 4.7).

The universe of discourse for X and Y are the real numbers from 0 to 20. For example, fuzzy subsets SMALL, MEDIUM, and LARGE for X can be represented in the for

$$\text{SMALL} = 1/0 + 0.91/1 + \cdots + 0.15/12 + 0/13 + \cdots + 0/19 + 0/20,$$

$$\text{MEDIUM} = 0/0 + 0/1 + \cdots + 0.8/12 + 0.95/13 + \cdots + 0.1/19 + 0/20,$$

$$\text{LARGE} = 0/0 + 0/1 + \cdots + 0/12 + 0/13 + \cdots + 0.9/19 + 1/20.$$

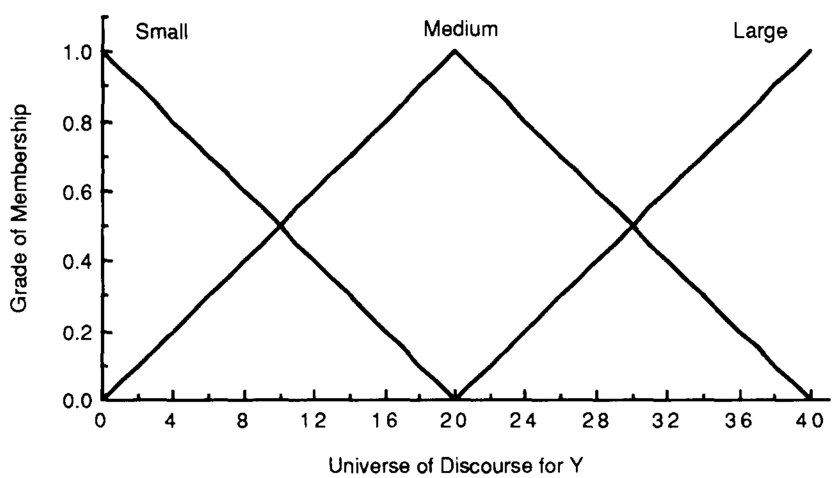

Fig. 4.3. The membership functions of fuzzy sets for Y on $10*\ln(1+x)/\ln(11)$.

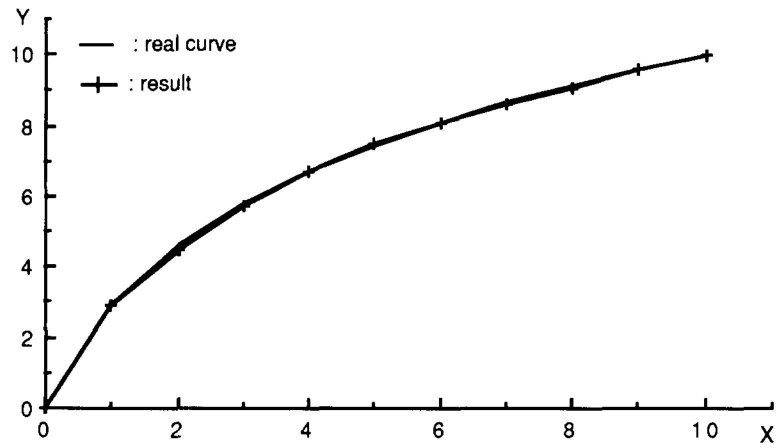

Fig. 4.4. Comparison between the actual curve and the result of fuzzy inference using R_5 on $10*\ln(1+x)/\ln(11)$.

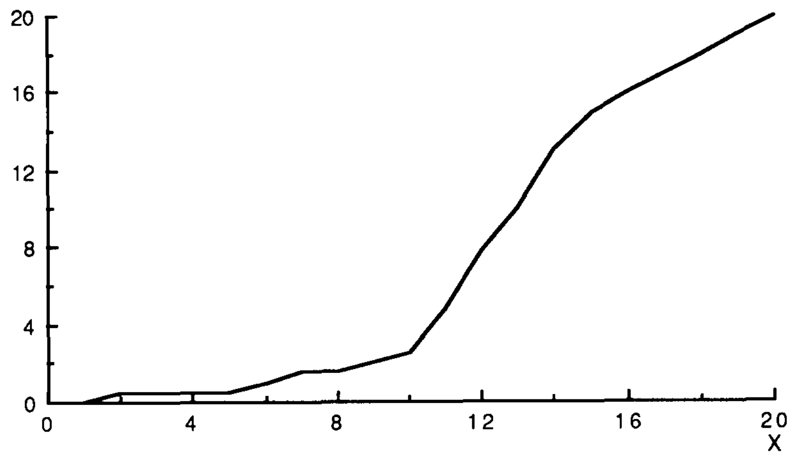

Fig. 4.5. A complicated curve.

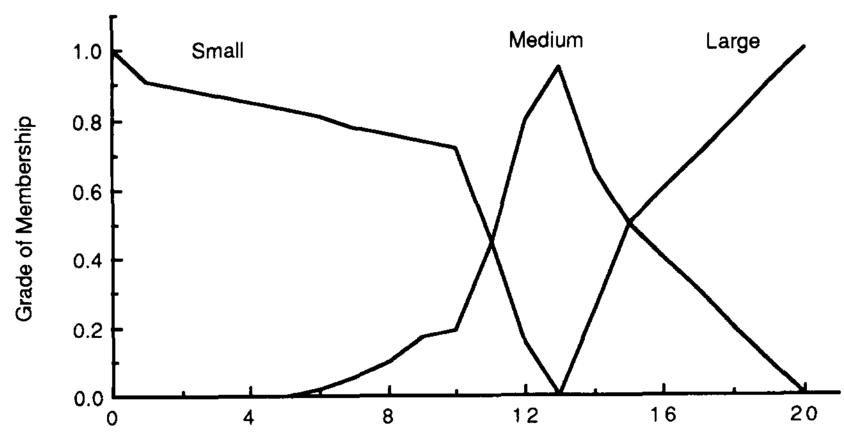

Fig. 4.6. The membership functions of fuzzy sets for X on complicated curve.

Table 4.2. Relation between X and Y on more complicated curve

X	Y	X	Y
0	0.00	10	2.25
1	0.00	11	4.75
2	0.50	12	7.75
3	0.50	13	10.00
4	0.50	14	13.00
5	0.50	15	15.00
6	1.00	16	16.00
7	1.50	17	17.00
8	1.50	18	18.00
9	2.00	19	19.00
		20	20.00

On the other hand, fuzzy sets SMALL, MEDIUM, and LARGE for Y can be represented in the forms:

$$\text{SMALL} = 1/0 + 0.9/1 + \cdots + 0/12 + 0/13 + \cdots + 0/19 + 0/20,$$

$$\text{MEDIUM} = 0/0 + 0.1/1 + \cdots + 1/10 + 0.9/11 + \cdots + 0.1/19 + 0/20,$$

$$\text{LARGE} = 0/0 + 0/1 + \cdots + 0.2/12 + 0.3/13 + \cdots + 0.9/19 + 1/20.$$

In order to achieve the correct result, we modify the membership functions of SMALL, MEDIUM, and LARGE for X (see Table 4.3 and Figure 4.6). However we fixed the membership functions of SMALL, MEDIUM, and LARGE for Y (see Figure 4.7).

Our result of the simulation is almost same as the real curve (see Figure 4.8).

Table 4.3. Fuzzy sets for X on more complicated curve

Universe X	Small	Medium	Large
0	1.00	0.00	0.00
1	0.91	0.00	0.00
2	0.89	0.00	0.00
3	0.87	0.00	0.00
4	0.85	0.00	0.00
5	0.83	0.00	0.00
6	0.81	0.02	0.00
7	0.78	0.05	0.00
8	0.76	0.10	0.00
9	0.74	0.17	0.00
10	0.72	0.19	0.00
11	0.45	0.45	0.00
12	0.15	0.80	0.00
13	0.00	0.95	0.00
14	0.00	0.65	0.25
15	0.00	0.50	0.50
16	0.00	0.40	0.60
17	0.00	0.30	0.70
18	0.00	0.20	0.80
19	0.00	0.10	0.90
20	0.00	0.00	1.00

We have shown that the accuracy of the simulation depends on the membership functions for X with three fixed verbal descriptions such as SMALL, MEDIUM, and LARGE and fixed membership functions for Y. However, it is interesting that the membership functions for X have some flexibility. As can be seen in Figure 4.9, the membership functions for X have upper limit and lower limits. For each of the membership functions for SMALL, MEDIUM, and LARGE, the upper line is the upper limit of the membership

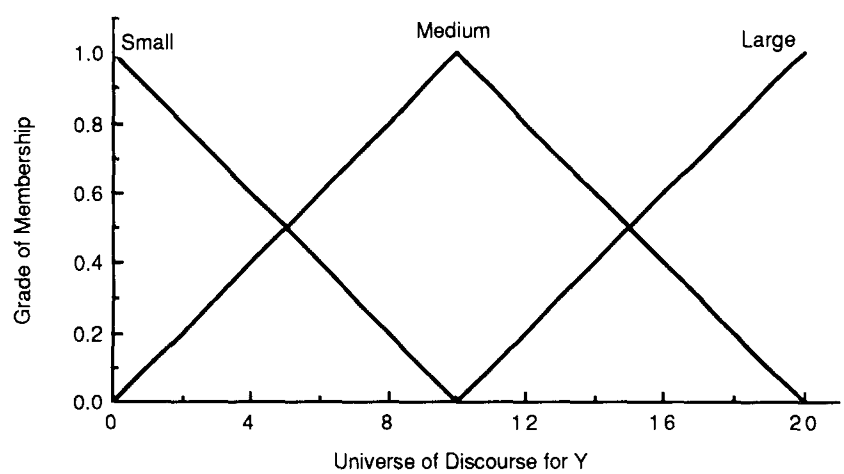

Fig. 4.7. The membership functions of fuzzy sets for Y on complicated curve.

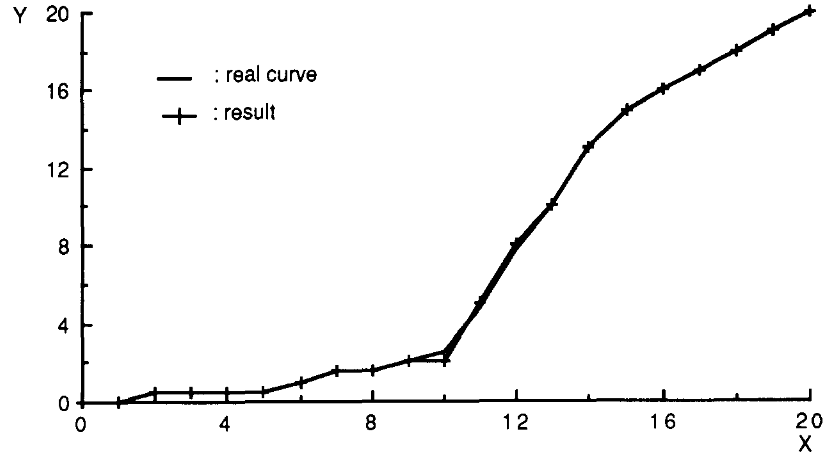

Fig. 4.8. Comparison between the actual curve and the result of fuzzy inference using R_5 on the complicated curve.

functions and the lower line is the lower limit. Namely, the membership functions can be taken at any point between the upper limit and lower limit. For example, when $x = 0$, the complicated curve has its value 0. For the vector $(x_{0,0}, x_{0,1}, x_{0,2})$, the grades of membership are $0.9 < x_{0,0} \leq 1$, $x_{0,1} = 0$, $x_{0,2} = 0$. When $x = 2$, the complicated curve has its value 0.5. For the vector $(x_{2,0}, x_{2,1}, x_{2,2})$, the grades of membership are $0.8 < x_{2,0} \leq 0.9$, $x_{2,1} = 0$, $x_{2,2} = 0$, etc.

4.3. New fuzzy implication operators

In this subsection, we introduce several new fuzzy implication operators such as:

$$R_a = \int_{U \times V} [1 \wedge (1 - (\mu_A(u) - \mu_B(u))^2)]/(u, v),$$

and

$$R_b = \int_{U \times V} [1 \wedge (1 - |\mu_A(u) - \mu_B(v)|)]/(u, v).$$

As with R_5, the sentence connective ALSO from verbal description for R_a and R_b is interpreted as intersection ($\wedge$ or minimum). The performance of these three implication operators is very good in the functional relation $Y = X$. We have exactly the same result as $Y = X$ using these fuzzy implication operators with only three verbal descriptions; SMALL, MEDIUM, and LARGE. In addition, the result is not affected even if the membership functions of X and Y are slightly different and more verbal descriptions such as "IF X is not small THEN Y is not small" are added to our original three verbal descriptions

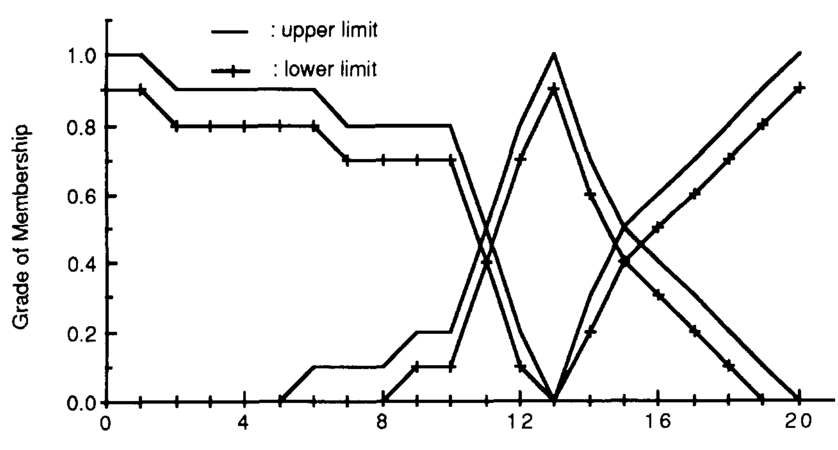

Fig. 4.9. The flexible membership functions of fuzzy sets for X on the complicated curve.

to describe the functional relation $Y = X$. Moreover, for the complicated curve, these implication operators including R_5 have very good results. Therefore, after our investigation, we suggest these implication operators as well as R_5.

5. Conclusion

This work is an extension of [6] where the applicability of the seventy-two implication operators with various membership functions in several typical cases were investigated. The five operators R_5, R_8, R_{22}, R_{25}, and R_{31} were shown to have very good performance characteristics in fuzzy inference.

In this paper, as an example, we have chosen the simplest functional relation $Y = X$. Based on previous research [6], we have further shown that the four operators R_8, R_{22}, R_{25}, and R_{31} do not have a good applicability for the case $Y = X$. However, R_5 has very good performance all the time in $Y = X$. We have show further the applicability of the fuzzy implication operator R_5 on $10 * \ln(1 + x)/\ln(11)$ as well as a more complicated curve. Furthermore, we have introduced the new fuzzy implication operators R_a and R_b which have excellent performance in all cases discussed in this paper.

References

[1] J.F. Baldwin, A new approach to approximate reasoning using a fuzzy logic, *Fuzzy Sets and Systems* **2** (1979) 309–325.

[2] J.F. Baldwin, Fuzzy logic and its application to fuzzy reasoning, in: M.M. Gupta et al., Eds., *Advances in Fuzzy Set Theory and Applications* (North-Holland, Amsterdam, 1979) 93–115.

[3] W. Bandler and L. Kohout, Fuzzy power sets and fuzzy implication operators, *Fuzzy Sets and Systems* **4** (1980) 13–30.

[4] W. Bandler and L. Kohout, Semantics of implication operators and fuzzy relational products, *Internat. J. Man–Machine Stud.* **12** (1980) 89–116.

[5] Z. Cao, K.H. Kim and F.W. Roush, *Incline Algebra and Applications* (Ellis Horwood, Homewood, IL, 1984).

[6] Z. Cao and A. Kandel, Applicability of some fuzzy implication operators, *Fuzzy Sets and Systems* **31** (1989) 151–186.

[7] Z. Cao, A. Kandel and L. Li, A new model of fuzzy reasoning, *Fuzzy Sets and Systems* **36** (1990) 311–325.

[8] S. Fukami, M. Mizumoto and K. Tanaka, Some considerations on fuzzy conditional inference, *Fuzzy Sets and Systems* **4** (1980) 243–273.

[9] A. Kandel, *Fuzzy Expert Systems* (CRC Press, Boca Raton, FL, 1992).

[10] A. Kandel, and G. Langholz, *Hybrid Architectures for Intelligent Systems* (CRC Press, Boca Raton, FL, 1992).

[11] J.B. Kiszka, M.E. Kochanska and D.S. Sliwinska, The influence of some fuzzy implication operators on the accuracy of a fuzzy model – Part I, *Fuzzy Sets and Systems* **15** (1985) 111–128.

[12] J.B. Kiszka, M.E. Kochanska and D.S. Sliwinska, The influence of some fuzzy implication operators on the accuracy of a fuzzy model – Part II, *Fuzzy Sets and Systems* **15** (1985) 223–240.

[13] E.H. Mamdani, Application for fuzzy algorithms for the control of a dynamic plant, *IEEE Proc.* **121** (1974) 1585–1588.

[14] E.H. Mamdani and S. Assilian, An experiment in synthesis with a fuzzy logic controller, *Internat. J. Man–Machine Stud.* **7** (1975) 1–13.

[15] E.H. Mamdani, Application of fuzzy logic to approximate reasoning using linguistic synthesis, *IEEE Trans. Comput.* **26** (1977) 1182–1191.

[16] M. Mizumoto, Fuzzy sets and their operations II, *Inform. and Control.* **50** (1981) 160–174.

[17] M. Mizumoto, Note on the arithmetic rule by Zadeh for fuzzy conditional inference, *Cybernetics and Systems* **12** (1981) 247–306.

[18] M. Mizumoto and H.-J. Zimmermann, Comparison of fuzzy reasoning methods, *Fuzzy Sets and Systems* **8** (1982) 253–283.

[19] M. Mizumoto, Comparison of various fuzzy reasoning method, in: *Preprints of Second IFSA Congress*, Tokyo (July 1987) 2–7.

[20] M. Mizumoto, Fuzzy controls under various approximate reasoning methods, in: *Preprints of Second IFSA Congress*, Tokyo (July 1987) 143–146.

[21] Y. Tsukamoto, An approach to fuzzy reasoning method, in: M.M. Gupta et al., Eds., *Advances in Fuzzy Set Theory and Applications* (North-Holland, Amsterdam, 1979) 137–149.

[22] R.R. Yager, An approach to inference in approximate reasoning, *Internat. J. Man–Machine Stud.* **13** (1980) 323–338.

[23] L.A. Zadeh, Similarity relations and fuzzy orderings, *Inform. Sci.* **3** (1971) 177–200.

[24] L.A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, *IEEE Trans. Systems Man Cybernet.* **3** (1973) 28–44.

[25] L.A. Zadeh, The concept of a linguistic and its applications to approximate reasoning I, II, III, *Inform. Sci.* **8** (1975) 199–249; 301–357; **9** (1975) 43–80.

[26] L.A. Zadeh, Fuzzy logic and approximate reasoning, *Synthese* **30** (1975) 407–428.

[27] L.A. Zadeh, A theory of approximate reasoning, in: J.E. Hayes et al., Eds., *Machine Intelligence* (John Wiley and Sons, New York, 1979) 149–194.